F1 June 2017 (MA)

Q1) $x^2 - \frac{5}{3}x + \frac{1}{3} = 0$

$\therefore \alpha\beta = \frac{1}{3}$ and $\alpha + \beta = \frac{5}{3}$

$\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$= \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{\left(\frac{5}{3}\right)^2 - 2\left(\frac{1}{3}\right)}{\frac{1}{3}}$

$= \boxed{\frac{19}{3}}$

Q2a) $AB = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -k & 2k \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 6-k-6 & 12+2k \\ -2+15 & -4 \end{pmatrix}$

2×3 3×2

$= \boxed{\begin{pmatrix} -k & 2k+12 \\ 13 & -4 \end{pmatrix}}$

b) $\det AB = (-k)(-4) - 13(2k+12) = 0$

$\Rightarrow 4k - 26k - 156 = 0$

$\Rightarrow 22k = -156$

$\Rightarrow k = \boxed{\frac{-78}{11}}$

prove: $\sum_{1}^{k+1} \frac{2}{r(r+1)(r+2)} = \frac{1}{2} - \frac{1}{(k+2)(k+3)}$

Q3a) $\underline{n=1}$: LHS $= \frac{2}{1(2)(3)} = \frac{1}{3}$ = RHS $= \frac{1}{2} - \frac{1}{6}$

$= 1/3$ //

$\therefore$ true for $n=1$.

<u>assume true for $n=k$</u>

(ie $\sum_{1}^{k} \frac{2}{r(r+1)(r+2)} = \frac{1}{2} - \frac{1}{(k+1)(k+2)}$)

<u>consider $n=k+1$</u>,

$$\sum_{1}^{k+1} \frac{2}{r(r+1)(r+2)} = \frac{1}{2} - \frac{1}{(k+1)(k+2)} + \frac{2}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{2} + \frac{2 - (k+3)}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{2} + \frac{2-3-k}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{2} + \frac{-k-1}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{2} - \frac{\cancel{(k+1)}}{\cancel{(k+1)}(k+2)(k+3)}$$

$$= \frac{1}{2} - \frac{1}{(k+2)(k+3)}$$

$\therefore$ true for $n=k+1$.

so, true for $n=1$.
true for $n=k+1$ when assumed true for $n=k$.
$\therefore$ By Mathematical Induction true for all $n \in \mathbb{Z}^+$

Q4a) $x = 4t$ $\quad y = \frac{4}{t}$

$\Rightarrow 3\left(\frac{4}{t}\right) - 2(4t) = 10$

$\Rightarrow \frac{12}{t} - 8t - 10 = 0$

$\times t$

$\Rightarrow 12 - 8t^2 - 10t = 0$

$\Rightarrow 8t^2 + 10t - 12 = 0$

By Quadratic formula, $t = \frac{3}{4}$, $t = -2$

$t = \frac{3}{4}$: $\boxed{\left[3, \frac{16}{3}\right]}$ $\quad t = -2$: $\boxed{\left[-8, -2\right]}$

b) midpoint : $\left(\frac{3-8}{2}, \frac{\frac{16}{3} - 2}{2}\right)$

$\Rightarrow \boxed{\left(\frac{-5}{2}, \frac{5}{3}\right)}$

Q5a) $f(2) = 2.94975\ldots$
$f(2.1) = -6.01055\ldots$

a	f(a)	b	f(b)	$\frac{a+b}{2}$	$f\left(\frac{a+b}{2}\right)$
2	2.9498..	2.1	-6.01055..	2.05	-1.31665..
2	2.9498...	2.05	-1.31665..	2.025	0.86847..

interval: $[2.025, 2.050]$

b) $f'(x) = -\frac{7}{2}x^{-\frac{3}{2}} - 5x^4$

$f'(2) = -81.2374\ldots$
$f(2) = 2.94975\ldots$

$\therefore x_1 = 2 - \frac{2.94975\ldots}{-81.23744\ldots} = 2.036\ldots$

$= \boxed{2.04}$

Q6a) $\sum_{r=1}^{n} r^2(r+1) = \sum_{r=1}^{n} r^3 + r^2 = \sum_{1}^{n} r^3 + \sum_{1}^{n} r^2$

$= \frac{n^2}{4}(n+1)^2 + \frac{1}{6}n(n+1)(2n+1)$

$= \frac{n}{12}(n+1)\left[3n(n+1) + 2(2n+1)\right]$

$= \frac{n}{12}(n+1)\left[3n^2 + 3n + 4n + 2\right]$

$= \frac{n}{12}(n+1)(3n^2 + 7n + 2)$

$$= \boxed{\frac{n}{12}(n+1)(n+2)(3n+1)}$$

b) $\sum_{1}^{49}[r^2(r+1)+2] - \sum_{1}^{24}[r^2(r+1)+2]$

$$= \sum_{1}^{49} r^2(r+1) + 2\sum_{1}^{49}(1) - \sum_{1}^{24} r^2(r+1) - 2\sum_{1}^{24}(1)$$

$$= \frac{49}{12}(50)(51)(3(49)+1) + 2(49) - \frac{24}{12}(25)(26)(3(24)+1) - 48$$

$$= 1541050 + 98 - 94900 - 48$$

$$= \boxed{1446200}$$

Q7a) $1-2i$ (complex conjugate pair)

bi) $(z-(1-2i))(z-(1+2i))(z^2+bz+d) = 0$

$$(z^2 - z(1+2i+1-2i) + (1+2i)(1-2i))(z^2+bz+d) = 0$$

$$(z^2-2z+5)(z^2+bz+d) = 0$$

$$z^4 + bz^3 + dz^2 - 2z^3 - 2bz^2 - 2dz + 5z^2 + 5bz + 5d = 0$$

$$z^4 + (b-2)z^3 + (d-2b+5)z^2 + (5b-2d)z + 5d = 0$$

Compare coefficients with given eqn.

z^3: $b-2 = 4$

$b = 6$

z: $5b - 2d = 4$

$\frac{30-4}{2} = d = 13$

constant: $a = 5d = \boxed{65}$

$\Rightarrow z^2 + 6z + 13 = 0$

$\left.\begin{array}{l} a=1 \\ b=6 \\ c=13 \end{array}\right\} z = \frac{-6 \pm \sqrt{36-4(13)}}{2} = \frac{-6 \pm 4i}{2}$

$= -3 \pm 2i$

So all roots are:

$$\boxed{\begin{array}{l} z = 1+2i \\ z = 1-2i \\ z = -3+2i \\ z = -3-2i \end{array}}$$

Q8a) $y^2 = 36x$

$2y\frac{dy}{dx} = 36$ (IMPLICITLY)

$\therefore \frac{dy}{dx} = \frac{18}{y} = \frac{18}{18p} = \frac{1}{p}$ //

$\Rightarrow y - 18p = \frac{1}{p}(x - 9p^2)$

$\Rightarrow y = \frac{1}{p}x - 9p + 18p$

$\Rightarrow y = \frac{1}{p}x + 9p$

$\xrightarrow{\times p} py = x + 9p^2$

$\Rightarrow py - x = 9p^2$ ◻

b) $y^2 = \boxed{36}x$

$\downarrow$

$4a = 36$

$\therefore a = \boxed{9}$ //

c) substitute $x = -9$, $y = 6$ into tangent :

$6p - -9 = 9p^2$

$9p^2 - 6p - 9 = 0$ //

$$3p^2 - 2p - 3 = 0$$

By Quadratic formula, $p = \dfrac{1 \pm \sqrt{10}}{3}$.

$$p > 0 \quad \therefore \; p = \boxed{\dfrac{1+\sqrt{10}}{3}}$$

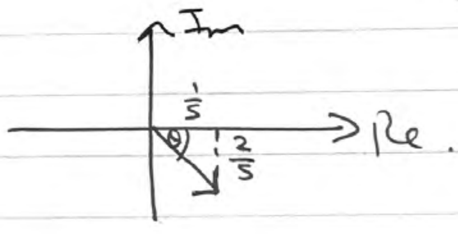
d) $9p^2 = \dfrac{9}{9}\left(1+\sqrt{10}\right) = 11 + 2\sqrt{10}$ //

$$18p = 6\left(1+\sqrt{10}\right)$$

$$\therefore \boxed{P \left(11 + 2\sqrt{10},\; 6\left(1+\sqrt{10}\right)\right)}$$

Q9a) $|z| = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \boxed{\dfrac{\sqrt{5}}{5}}$

$$\arg z = -\left(\tan^{-1}(2)\right)$$
$$= \boxed{-1.11^c}$$



$\tan\theta = 2$

b) $w = \dfrac{\lambda i}{z}$

$$w = \frac{\lambda i}{\frac{1}{5} - \frac{2}{5}i} = \frac{5\lambda i}{1-2i}$$

$$w = \frac{5\lambda i \times (1+2i)}{(1-2i)\times(1+2i)} = \frac{5\lambda i + 10\lambda i^2}{1+4}$$

$$= \frac{5\lambda i - 10\lambda}{5}$$

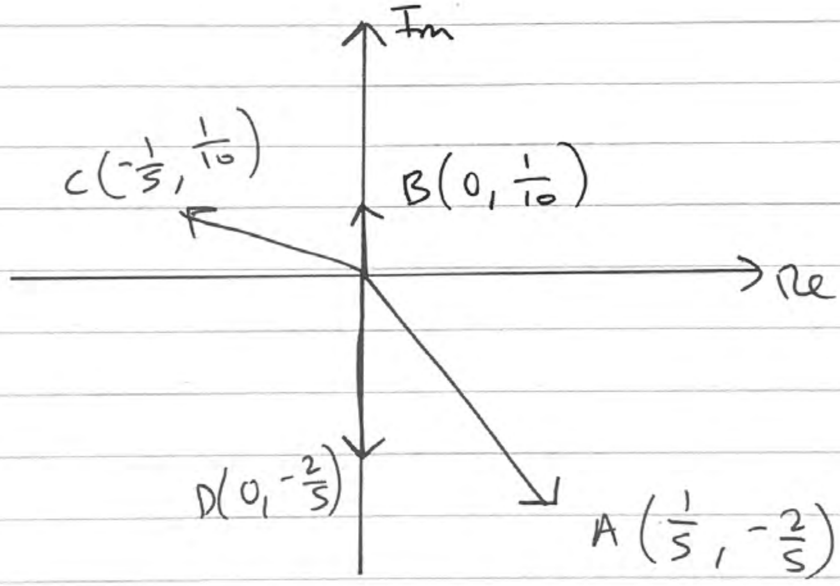
$$= \boxed{-2\lambda + \lambda i}$$

c i) $\lambda = \frac{1}{10}$: $w = -\frac{2}{10} + \frac{1}{10} i$ //

$\therefore \frac{4}{3}(z+w) = \frac{4}{3}\left(\frac{1}{5} - \frac{1}{5} + \frac{1}{10} i - \frac{4}{10} i\right)$

$= \frac{4}{3}\left(\frac{-3}{10} i\right) = \boxed{-\frac{2}{5} i}$

ii)



formula booklet

Q10a) $\begin{pmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}}$

b) $\begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}}$

c) $R = VU = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

$= \boxed{\begin{pmatrix} \frac{\sqrt{2}-\sqrt{6}}{4} & \frac{-\sqrt{2}-\sqrt{6}}{4} \\ \frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \end{pmatrix}}$

d) R is in the same form as a matrix that represents an anticlockwise rotation about (0, 0).

$\left.\begin{aligned} \sin\theta &= \frac{\sqrt{2}+\sqrt{6}}{4} \\ \cos\theta &= \frac{\sqrt{2}-\sqrt{6}}{4} \end{aligned}\right\} \theta = 105^\circ \quad \therefore$ $\boxed{\text{Anticlockwise rotation of } 105^\circ \text{ about } (0, 0)}$

($\theta = 105^\circ$ fits both equations)

$$\sin 105 = \frac{\sqrt{2}+\sqrt{6}}{4} = \sin(180-105) = \sin 75 \ //$$

$$\therefore \sin 75 = \frac{(\sqrt{2}+\sqrt{6}) \times \frac{\sqrt{2}}{2}}{(4) \times \frac{\sqrt{2}}{2}}$$

$$= \frac{1 + \frac{\sqrt{12}}{2}}{2\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

$$\square$$

$$\cos 105 = \frac{\sqrt{2}-\sqrt{6}}{4} = -\cos(180-105) = -\cos 75$$

$$= -\frac{(\sqrt{2}-\sqrt{6}) \times \frac{\sqrt{2}}{2}}{(4) \times \frac{\sqrt{2}}{2}}$$

$$= -\frac{\left(1 - \frac{\sqrt{12}}{2}\right)}{2\sqrt{2}} = -\frac{(1-\sqrt{3})}{2\sqrt{2}}$$

$$= \boxed{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$